

Singularities in Fully Developed Turbulence

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Abstract

Phenomenological arguments are used to explore finite-time singularity development in several fully developed turbulence (FDT) situations. The role played by the cascade physics underlying this process is investigated. Such diverse aspects as the effects of spatial intermittency and fluid compressibility in three-dimensional (3D) and the divorticity amplification mechanism in two-dimensional (2D) and the advection-diffusion mechanism in magnetohydrodynamic turbulence are considered to provide physical insights into this process in variant cascade physics situations.

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1. Introduction

The existence of strongly localized features like vorticity sheets in the small-scale structure of fully-developed turbulence (FDT) suggests the development of singularities in the flow velocity field (Gibbon [1]). It is generally believed that an understanding of this process¹ is crucial to the development of a viable theory of turbulence (Constantin [2]). A rigorous result due to Beale et al. [3] (see also Majda and Bertozzi [4]) requires the magnitude of the vorticity to become infinite to allow the occurrence of a finite-time singularity. Numerical investigations of three-dimensional (3D) FDT have suggested but failed to provide a conclusive evidence that ideal-flow solutions, starting from regular initial conditions, will spontaneously develop a singularity in *finite* time (Brachet et al. [5]).

Phenomenological considerations, which predict development of finite-time singularities in FDT, are believed to over-estimate the nonlinear effects (Frisch [6]) because nonlinearity depletion mechanisms via coherent structure generation seem to be operational (Constantin [2], Frisch [6]). Nonetheless, here we propose to use phenomenological arguments to explore finite-time singularity development in several FDT situations because, from a qualitative point of view, such considerations do seem to be able to provide useful physical insights into this process in variant cascade physics situations.

2. 3D Incompressible FDT

The energy dissipation rate in 3D FDT is given by (in usual notation),

$$\varepsilon \sim \nu \frac{v^2}{\eta^2}. \quad (1)$$

η being the Kolmogorov microscale. On using the Kolmogorov scaling,

$$v \sim \varepsilon^{1/3} \eta^{1/3} \quad (2)$$

(1) gives the well-known result,

$$\eta \sim \frac{\nu^{3/4}}{\varepsilon^{3/4}}. \quad (3)$$

On using (3), the vorticity evolution equation,

$$\frac{d\omega}{dt} = \nu \nabla^2 \omega \sim \nu \frac{\omega}{\eta^2} \quad (4)$$

leads to

$$\frac{d\omega}{dt} \sim \frac{\varepsilon^{1/2}}{\nu^{1/2}} \omega. \quad (5)$$

Invoking the *dissipative anomaly*,

$$\varepsilon \sim \omega^2 \nu \sim \text{const} \quad (6)$$

¹One of the issues in this regard is the connection, if any, between the *dissipative anomaly* in FDT and the finite-time singularity.

(5) leads to

$$\frac{d\omega}{dt} \sim \omega^2 \quad (7)$$

and hence the well-known result,

$$\omega \sim \frac{1}{t+c} \quad (8)$$

suggesting a finite-time singularity.

3. Effects of Spatial Intermittency in 3D FDT

Spatial intermittency effects associated with the spatial spottiness of the turbulent activity become more pronounced at small scales. So, one may surmise spatial intermittency to have a significant effect on the finite-singularity development. One may incorporate spatial intermittency effects via the fractal nature of dissipative structures where the turbulent activity is concentrated, following Mandelbrot [7]. In a first approximation, the dissipative structures may be approximated by a homogeneous fractal with a non-integer Hausdorff dimension D (Frisch et al. [8]).

Assuming the scaling behavior,

$$v(\eta) \sim \eta^\alpha \quad (9)$$

we have,

$$\varepsilon(\eta) \sim \eta^{3\alpha-1}. \quad (10)$$

Using (10), (3) gives (Paladin and Vulpiani [9]),

$$\eta \sim \nu^{\frac{1}{1+\alpha}}. \quad (11)$$

The homogeneous fractal model for the 3D FDT (Frisch et al. [8]) gives

$$\alpha = \frac{1}{3}(D-2). \quad (12)$$

Using (12), (11) gives

$$\eta \sim \nu^{\frac{3}{D+1}}. \quad (13)$$

Using (13), (4) leads to

$$\frac{d\omega}{dt} \sim \nu^{\frac{D-5}{D+1}} \omega. \quad (14)$$

Using (6), (14) becomes

$$\frac{d\omega}{dt} \sim \omega^{\frac{11-D}{D+1}} \quad (15)$$

from which,

$$\omega \sim (t+c)^{-1+\frac{3}{2}(\frac{D-3}{D-5})}. \quad (16)$$

The weakened finite-time singularity due to spatially intermittency ($D < 3$), indicated by (16), reflects a nonlinearly depletion activity occurring in the latter case via generation of coherent structures, as conjectured by Frisch [6].

4. Effects of Compressibility

In view of the intuitive belief that vortices tend to stretch stronger in a compressible fluid it is pertinent to explore the effect of fluid compressibility on the finite-time singularity development.

For the compressible case, the kinetic energy dissipation rate ε is given by

$$\hat{\varepsilon} \sim \frac{\rho v^3}{\hat{\eta}} \sim \mu \frac{v^2}{\hat{\eta}^2} \quad (17)$$

from which, we obtain

$$\hat{\eta} \sim \frac{\mu}{\rho v} \quad (18a)$$

$$v \sim \left(\frac{\hat{\varepsilon} \hat{\eta}}{\rho} \right)^{1/3} \quad (18b)$$

and hence (Shivamoggi [10]), the Kolmogorov microscale $\hat{\eta}$ for compressible turbulence is given by

$$\hat{\eta} \sim \left(\frac{\mu^3}{\rho^2 \hat{\varepsilon}} \right)^{1/4}. \quad (19)$$

The vorticity evolution equation

$$\rho \frac{d\omega}{dt} \sim \mu \frac{\omega}{\hat{\eta}^2} \quad (20)$$

on using (19), becomes

$$\frac{d\omega}{dt} \sim \frac{\hat{\varepsilon}^{1/2}}{\mu^{1/2}} \omega. \quad (21)$$

Noting the dissipative anomaly for the compressible case (Shivamoggi [11]),

$$\hat{\varepsilon} \sim \mu \omega^2 \sim \text{const} \quad (22)$$

(21) becomes

$$\frac{d\omega}{dt} \sim \omega^2 \quad (23)$$

and hence

$$\omega \sim \frac{1}{t + c} \quad (24)$$

as in the incompressible case. The apparent absence of a compressibility correction to (8) is probably due to the fact that compressibility effects are not length-scale dependent and materialize at all length scales.

5. Enstrophy Cascade in 2D FDT

As a further example of the effect of the underlying cascade physics on the finite-time singularity development, let us consider the enstrophy cascade in 2D FDT.

Noting that divorticity (Kida [12]) amplification constitutes the physical mechanism underlying the enstrophy cascade in 2D FDT (Kuznetsov et al. [13]) the divorticity \mathbf{b}

$$\mathbf{b} \equiv \nabla \times \boldsymbol{\omega} \quad (25)$$

appears to be the appropriate physical variable to characterize the dynamics in question (Shivamoggi et al. [14]).

The divorticity evolution equation

$$\frac{db}{dt} = \nu \nabla^2 b \sim \nu \frac{b}{\zeta^2} \quad (25)$$

on noting that the Kraichnan microscale ζ is given by (Shivamoggi [15]),

$$\zeta \sim \frac{\nu^{1/2}}{\tau^{1/6}} \quad (26)$$

τ being the enstrophy dissipation rate, leads to

$$\frac{db}{dt} \sim \tau^{1/3} b. \quad (27)$$

(27) further leads to

$$b \sim e^{\tau^{1/3} t} \quad (28)$$

confirming the well known absence of a finite-time singularity in 2D FDT (Rose and Sulem [16]).

6. Magnetohydrodynamic Turbulence

In view of the advection-diffusion mechanism that controls the statistical properties of magnetic field² the magnetohydrodynamic (MHD) turbulence (Shivamoggi [19]) presents a convenient framework to explore the effect of the advection-diffusion mechanism on the finite-time singularity development. Here, we adopt the Iroshnikov [20] - Kraichnan [21] (IK) phenomenology.

The energy dissipation rate is given by

$$\varepsilon \sim \frac{v^4}{\hat{\eta} C_A} \sim \eta_m \frac{v^2}{\hat{\eta}^2} \quad (29)$$

from which, we obtain

$$\hat{\eta} \sim \frac{\eta_m C_A}{v^2} \quad (30a)$$

$$v \sim \varepsilon^{1/4} C_A^{1/4} \hat{\eta}^{1/4} \quad (30b)$$

²Advective stretching of magnetic field lines leads to amplification of magnetic field (Batchelor [17]) while magnetic field lines that have been highly stretched typically experience stronger Ohmic dissipation (Kida et al. [18]).

η_m being the magnetic resistivity, and C_A being the velocity of Alfvén waves in the magnetic field of the large-scale eddies. Combining (30a, b), the Kolmogorov microscale $\hat{\eta}$ for MHD turbulence is given by

$$\hat{\eta} \sim \left(\frac{\eta_m^2 C_A}{\varepsilon} \right)^{1/3}. \quad (31)$$

The current density J evolution equation

$$\frac{dJ}{dt} \sim \eta_m \frac{J}{\hat{\eta}^2} \quad (32)$$

on using (31), becomes

$$\frac{dJ}{dt} \sim \frac{\varepsilon^{2/3}}{C_A^{2/3} \eta_m^{1/3}} J. \quad (33)$$

Noting the dissipative anomaly for MHD turbulence (Shivamoggi [19]),

$$\varepsilon \sim \eta_m J^2 \sim \text{const} \quad (34)$$

(33) becomes

$$\frac{dJ}{dt} \sim \frac{\varepsilon^{1/3}}{C_A^{2/3}} J^{5/3}. \quad (35)$$

(35) leads to

$$J \sim \frac{1}{(t + c)^{3/2}} \quad (36)$$

showing a weakened finite-time singularity in the IK phenomenology, which is plausible because the latter is known (Shivamoggi [22]) to correspond to a *weakly-nonlinear* regime.

7. Discussion

In this paper phenomenological arguments have been used to provide some insights into the finite-time singularity development in several FDT situations. Particular attention is paid to the role played by the cascade physics underlying this process. Such diverse aspects as the effects of spatial intermittency and fluid compressibility in 3D and the divorticity amplification mechanism in 2D and the advection-diffusion mechanism in MHD turbulence are considered to gain better physical understanding of this process.

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